

# Possible formulations for three-charged particles correlations in terms of Coulomb wave functions

T. Mizoguchi,<sup>a,1</sup> M. Biyajima<sup>b,2</sup>

<sup>a</sup>*Toba National College of Maritime Technology, Toba 517-8501, Japan*

<sup>b</sup>*Department of Physics, Faculty of Science, Shinshu University, Matsumoto 390-8621, Japan*

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## Abstract

The recent data for Bose-Einstein Correlations (BEC) of three-charged particles obtained by NA44 Collaboration have been analysed using theoretical formula with Coulomb wave functions. It has been recently proposed by Alt et al. It turns out that there are discrepancies between these data and the respective theoretical values. To resolve this problem we seek a possibly modified theoretical formulation of this problem by introducing the degree of coherence for the exchange effect due to the BEC between two-identical bosons. As a result we obtain a modified formulation for the BEC of three-charged particles showing good agreement with the data. Moreover, we investigate physical connection between our modified formulation and the core-halo model proposed by Csörgő et al. Our study indicates that the interaction region estimated by the BEC of three-charged particles in the S + Pb collisions at 200 GeV/c per nucleon is equal to about 1.5 fm~1.8 fm.

*Key words:* Bose-Einstein Correlation, three-charged particles, Coulomb wave functions, high energy heavy-ion collisions

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## 1 Introduction

One of the most interesting subjects in high energy heavy-ion collisions is study of the higher order Bose-Einstein Correlation (BEC) effect [1–6] (known also as the HBT or the GGLP effect, or as the hadron interferometry [7–10]). From

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<sup>1</sup> E-mail: mizoguti@toba-cmt.ac.jp

<sup>2</sup> E-mail: biyajima@azusa.shinshu-u.ac.jp

data on BEC we can (in principle) infer the size of the interaction region and therefore estimate the energy densities reached in high energy collisions. Such work is a necessary task in the search for the quark-gluon plasma [11,12] - a new, hypothetical form of matter.

To get more precise sizes of the interaction regions, we have to take into account the final state interactions among the charged particles [13,14]. A great advance in this direction for the BEC of the three-charged particles has been recently made by Alt et. al. [5]. They have derived a correction formula for the raw data introducing distribution functions of the charged particles. Their formulation is based on the plane wave functions and on the Coulomb wave functions, assuming that produced hadrons are already in the asymptotic region of the Coulomb interactions where the strong interaction already vanishes [15,16]. It amounts in the following correction factor  $K_{Coul}$  due to the Coulomb effect for identical three-charged particles: <sup>3</sup>

$$K_{Coul} = \frac{N_{Coul}}{D_{plane}} . \quad (1)$$

The denominator  $D_{plane}$  is given by ( $\rho(\mathbf{x}_i)$  are distribution functions of charged particles):

$$\begin{aligned} D_{plane} \cong & \frac{1}{6} \int d^3\mathbf{x}_1 \rho(\mathbf{x}_1) d^3\mathbf{x}_2 \rho(\mathbf{x}_2) d^3\mathbf{x}_3 \rho(\mathbf{x}_3) \\ & \cdot \left| e^{i(\mathbf{k}_1 \cdot \mathbf{x}_1 + \mathbf{k}_2 \cdot \mathbf{x}_2 + \mathbf{k}_3 \cdot \mathbf{x}_3)} + e^{i(\mathbf{k}_1 \cdot \mathbf{x}_2 + \mathbf{k}_2 \cdot \mathbf{x}_1 + \mathbf{k}_3 \cdot \mathbf{x}_3)} \right. \\ & + e^{i(\mathbf{k}_1 \cdot \mathbf{x}_2 + \mathbf{k}_2 \cdot \mathbf{x}_3 + \mathbf{k}_3 \cdot \mathbf{x}_1)} + e^{i(\mathbf{k}_1 \cdot \mathbf{x}_1 + \mathbf{k}_2 \cdot \mathbf{x}_3 + \mathbf{k}_3 \cdot \mathbf{x}_2)} \\ & \left. + e^{i(\mathbf{k}_1 \cdot \mathbf{x}_3 + \mathbf{k}_2 \cdot \mathbf{x}_1 + \mathbf{k}_3 \cdot \mathbf{x}_2)} + e^{i(\mathbf{k}_1 \cdot \mathbf{x}_3 + \mathbf{k}_2 \cdot \mathbf{x}_2 + \mathbf{k}_3 \cdot \mathbf{x}_1)} \right|^2 , \end{aligned} \quad (2)$$

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<sup>3</sup> The correlation functions for two and three-identical particles are given as usual by

$$\frac{N^{(2+ \text{ or } 2-)}}{N^{BG}} = \frac{P(\mathbf{k}_1, \mathbf{k}_2)}{P(\mathbf{k}_1)P(\mathbf{k}_2)} \quad \text{and} \quad \frac{N^{(3+ \text{ or } 3-)}}{N^{BG}} = \frac{P(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)}{P(\mathbf{k}_1)P(\mathbf{k}_2)P(\mathbf{k}_3)} ,$$

where  $\mathbf{k}_i$  is the momentum of particle  $i$ , and  $P(\mathbf{k}_1, \mathbf{k}_2)$  and  $P(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$  are two and three particles probability densities, respectively. The probability densities for two-identical particles case can be written as,

$$\int \int \left| \psi_{\mathbf{k}_1 \mathbf{k}_2}^{BE}(\mathbf{x}_1, \mathbf{x}_2) \right|^2 \rho(\mathbf{x}_1) \rho(\mathbf{x}_2) d^3\mathbf{x}_1 d^3\mathbf{x}_2 ,$$

where

$$\psi_{\mathbf{k}_1 \mathbf{k}_2}^{BE}(\mathbf{x}_1, \mathbf{x}_2) = \frac{1}{\sqrt{2}} [\psi_{\mathbf{k}_1 \mathbf{k}_2}(\mathbf{x}_1, \mathbf{x}_2) + \psi_{\mathbf{k}_1 \mathbf{k}_2}(\mathbf{x}_2, \mathbf{x}_1)] ,$$

$\rho(\mathbf{x}_i)$  stand for the source functions of particle  $i$ .

The numerator  $N_{Coul}$  has the following form:

$$\begin{aligned}
N_{Coul} \cong & \frac{1}{6} \int d^3\mathbf{x}_1 \rho(\mathbf{x}_1) d^3\mathbf{x}_2 \rho(\mathbf{x}_2) d^3\mathbf{x}_3 \rho(\mathbf{x}_3) \\
& \cdot \left| \psi_{\mathbf{k}_1\mathbf{k}_2}^C(\mathbf{x}_1, \mathbf{x}_2) \psi_{\mathbf{k}_2\mathbf{k}_3}^C(\mathbf{x}_2, \mathbf{x}_3) \psi_{\mathbf{k}_3\mathbf{k}_1}^C(\mathbf{x}_3, \mathbf{x}_1) \right. \\
& + \psi_{\mathbf{k}_1\mathbf{k}_2}^C(\mathbf{x}_1, \mathbf{x}_3) \psi_{\mathbf{k}_2\mathbf{k}_3}^C(\mathbf{x}_3, \mathbf{x}_2) \psi_{\mathbf{k}_3\mathbf{k}_1}^C(\mathbf{x}_2, \mathbf{x}_1) \\
& + \psi_{\mathbf{k}_1\mathbf{k}_2}^C(\mathbf{x}_2, \mathbf{x}_1) \psi_{\mathbf{k}_2\mathbf{k}_3}^C(\mathbf{x}_1, \mathbf{x}_3) \psi_{\mathbf{k}_3\mathbf{k}_1}^C(\mathbf{x}_3, \mathbf{x}_2) \\
& + \psi_{\mathbf{k}_1\mathbf{k}_2}^C(\mathbf{x}_2, \mathbf{x}_3) \psi_{\mathbf{k}_2\mathbf{k}_3}^C(\mathbf{x}_3, \mathbf{x}_1) \psi_{\mathbf{k}_3\mathbf{k}_1}^C(\mathbf{x}_1, \mathbf{x}_2) \\
& + \psi_{\mathbf{k}_1\mathbf{k}_2}^C(\mathbf{x}_3, \mathbf{x}_1) \psi_{\mathbf{k}_2\mathbf{k}_3}^C(\mathbf{x}_1, \mathbf{x}_2) \psi_{\mathbf{k}_3\mathbf{k}_1}^C(\mathbf{x}_2, \mathbf{x}_3) \\
& \left. + \psi_{\mathbf{k}_1\mathbf{k}_2}^C(\mathbf{x}_3, \mathbf{x}_2) \psi_{\mathbf{k}_2\mathbf{k}_3}^C(\mathbf{x}_2, \mathbf{x}_1) \psi_{\mathbf{k}_3\mathbf{k}_1}^C(\mathbf{x}_1, \mathbf{x}_3) \right|^2 . \tag{3}
\end{aligned}$$

Here  $\psi_{\mathbf{k}_i\mathbf{k}_j}^C(\mathbf{x}_i, \mathbf{x}_j)$  are the Coulomb wave functions of the respective 2-body collision expressed as,

$$\psi_{\mathbf{k}_i\mathbf{k}_j}^C(\mathbf{x}_i, \mathbf{x}_j) = \Gamma(1 + i\eta_{ij}) e^{-\pi\eta_{ij}/2} e^{i\mathbf{k}_{ij} \cdot \mathbf{r}_{ij}} F[-i\eta_{ij}, 1; i(k_{ij}r_{ij} - \mathbf{k}_{ij} \cdot \mathbf{r}_{ij})], \tag{4}$$

with  $\mathbf{r}_{ij} = (\mathbf{x}_i - \mathbf{x}_j)$ ,  $\mathbf{k}_{ij} = (\mathbf{k}_i - \mathbf{k}_j)/2$ ,  $r_{ij} = |\mathbf{r}_{ij}|$ ,  $k_{ij} = |\mathbf{k}_{ij}|$  and  $\eta_{ij} = m\alpha/k_{ij}$ .  $F[a, b; x]$  and  $\Gamma(x)$  are the confluent hypergeometric function and the Gamma function, respectively. In order to use Eqs. (1), (2) and (3), one has to assume first some shapes and sizes for the source functions. In fact, this is the procedure already used in Ref. [17] by NA44 Collaboration:

$$\text{Corrected data} = (\text{raw data}) \times K_{\text{spc}} \times K_{\text{acceptance}} \times K_{Coul} ,$$

where  $K_{\text{spc}}$  and  $K_{\text{acceptance}}$  denote the effect of multiparticle production in the single particle spectra and the acceptance effect in the experiment.

In this paper, we would like to adopt a different point of view for Eq. (3). As is seen in Ref. [14], the BEC of identical two-charged pions can also be analysed by the Coulomb wave functions. It is therefore reasonable to expect that the numerator  $N_{Coul}$  is the main theoretical ingredient in analysis of the BEC of three-charged particles. We argue therefore that

$$N^{(3+ \text{ or } 3-)} / N^{BG} \equiv C \times N_{Coul} , \tag{5}$$

where we have introduced the normalization factor  $C$ , which corresponds to the asymptotic value of the BEC. Using Eq. (5) we can now (with the help of the CERN-MINUIT program) analyse data of Ref. [17] using Gaussian source distributions of radii  $R$ ,  $\rho(\mathbf{x}) = \frac{1}{(2\pi R^2)^{3/2}} \exp\left[-\frac{\mathbf{x}^2}{2R^2}\right]$ <sup>4</sup>.

<sup>4</sup> It should be remembered that NA44 Collaboration data are for the variable

$$Q_3^2(4D) = (k_1 - k_2)^2 + (k_2 - k_3)^2 + (k_3 - k_1)^2$$

In the next paragraph, we analyse the data of NA44 Collaboration [17] by Eq. (5). In the third paragraph we shall derive a modified theoretical formula for 3-particle BEC introducing the degree of coherence parameter into Eq. (5). This formula will be then used in the 4th paragraph for the re-analyses of the experimental data [17]. In the 5th paragraph, we investigate whether or not there is physical connection between our study and the core-halo model [18]. Concluding remarks are given in the final paragraph.

## 2 Application of Eq. (5) to the data by NA44 Collaboration

Here we analyse the data by Eq. (5). As can be seen in Fig. 1 and Table 1, there are some discrepancies between the data points and theoretical values calculated by means of Eq. (5). The minimum  $\chi^2$  value, 17.6 in Table 1, seems to be big, as the number of the data points are considered. Thus we would like to know why this equation cannot explain the data [17]. We know that there are several possible reasons due to effects of the partial coherence, the contamination [19] and the long-lived resonances [18]. At present, we consider the effect of the possible partial coherent of produced pions. In fact, authors of Ref. [17] have used not the equivalence of Eq. (5) but the following formula instead (cf., Ref. [9,20]):

$$\frac{N^{(3+)}}{N^{BG}} = C \left( 1 + \lambda_3 e^{-R_3^2 Q_3^2} \right) . \quad (6)$$

It contains one more parameter,  $\lambda_3$ , which can be regarded as a kind of effective degree of coherence and which, in our opinion, should therefore occur also somehow in Eq. (5).

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where  $k_i$  are four-momentum of charged particles.  $Q_3 = \sqrt{Q_3^2}$ . However, in our calculations we assume that  $q_{0,ij}^2 = (k_{0i} - k_{0j})^2 \approx 0$  and use, instead,

$$Q_3^2(3D) = (\mathbf{k}_1 - \mathbf{k}_2)^2 + (\mathbf{k}_2 - \mathbf{k}_3)^2 + (\mathbf{k}_3 - \mathbf{k}_1)^2 .$$

There is an approximation between them

$$Q_3(4D) \cong Q_3(3D) - \sum q_{0,ij}^2 / 2Q_3(3D) .$$

$Q_3(4D)$  depends on sum of squared energy differences.

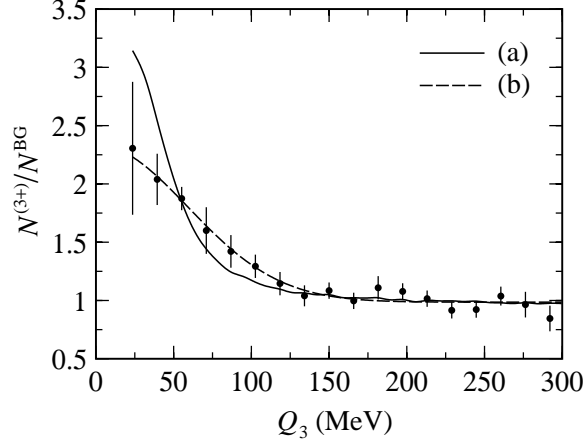


Fig. 1. Analysis of  $3\pi^+$  BEC in S + Pb collision [17]. (a) and (b) are results of Eqs. (5) and (6), respectively. The error bars are sum of statistical and systematic errors.

Table 1

Estimated values for the data [17] by Eqs. (5) and (6) using CERN-MINUIT program.

Formulas	$C$	$R$ [fm]	$\lambda_3$	$\chi^2/N_{dof}$
Eq. (5)	$0.941 \pm 0.026$	$2.47 \pm 0.14$	—	17.6/16
Eq. (6)	$0.986 \pm 0.028$	$2.36 \pm 0.26$	$1.37 \pm 0.19$	7.8/15

### 3 Diagram Decomposition of Eq. (5)

First of all, we have to find a possible way for the introduction of the degree of coherence parameter  $\lambda$  into Eq. (5). Let us therefore examine the plane wave (PW) approximations of the Coulomb wave functions:

$$A(1) = \psi_{\mathbf{k}_1\mathbf{k}_2}^C(\mathbf{x}_1, \mathbf{x}_2) \psi_{\mathbf{k}_2\mathbf{k}_3}^C(\mathbf{x}_2, \mathbf{x}_3) \psi_{\mathbf{k}_3\mathbf{k}_1}^C(\mathbf{x}_3, \mathbf{x}_1) \xrightarrow{\text{PW}} e^{i\mathbf{k}_{12} \cdot \mathbf{r}_{12}} e^{i\mathbf{k}_{23} \cdot \mathbf{r}_{23}} e^{i\mathbf{k}_{31} \cdot \mathbf{r}_{31}} = e^{(3/2)i(\mathbf{k}_1 \cdot \mathbf{x}_1 + \mathbf{k}_2 \cdot \mathbf{x}_2 + \mathbf{k}_3 \cdot \mathbf{x}_3)}, \quad (7a)$$

$$A(2) = \psi_{\mathbf{k}_1\mathbf{k}_2}^C(\mathbf{x}_1, \mathbf{x}_3) \psi_{\mathbf{k}_2\mathbf{k}_3}^C(\mathbf{x}_3, \mathbf{x}_2) \psi_{\mathbf{k}_3\mathbf{k}_1}^C(\mathbf{x}_2, \mathbf{x}_1) \xrightarrow{\text{PW}} e^{i\mathbf{k}_{12} \cdot \mathbf{r}_{13}} e^{i\mathbf{k}_{23} \cdot \mathbf{r}_{32}} e^{i\mathbf{k}_{31} \cdot \mathbf{r}_{21}} = e^{(3/2)i(\mathbf{k}_1 \cdot \mathbf{x}_1 + \mathbf{k}_2 \cdot \mathbf{x}_3 + \mathbf{k}_3 \cdot \mathbf{x}_2)}, \quad (7b)$$

$$A(3) = \psi_{\mathbf{k}_1\mathbf{k}_2}^C(\mathbf{x}_2, \mathbf{x}_1) \psi_{\mathbf{k}_2\mathbf{k}_3}^C(\mathbf{x}_1, \mathbf{x}_3) \psi_{\mathbf{k}_3\mathbf{k}_1}^C(\mathbf{x}_3, \mathbf{x}_2) \xrightarrow{\text{PW}} e^{i\mathbf{k}_{12} \cdot \mathbf{r}_{21}} e^{i\mathbf{k}_{23} \cdot \mathbf{r}_{13}} e^{i\mathbf{k}_{31} \cdot \mathbf{r}_{32}} = e^{(3/2)i(\mathbf{k}_1 \cdot \mathbf{x}_2 + \mathbf{k}_2 \cdot \mathbf{x}_1 + \mathbf{k}_3 \cdot \mathbf{x}_3)}, \quad (7c)$$

$$A(4) = \psi_{\mathbf{k}_1\mathbf{k}_2}^C(\mathbf{x}_2, \mathbf{x}_3) \psi_{\mathbf{k}_2\mathbf{k}_3}^C(\mathbf{x}_3, \mathbf{x}_1) \psi_{\mathbf{k}_3\mathbf{k}_1}^C(\mathbf{x}_1, \mathbf{x}_2) \xrightarrow{\text{PW}} e^{i\mathbf{k}_{12} \cdot \mathbf{r}_{23}} e^{i\mathbf{k}_{23} \cdot \mathbf{r}_{31}} e^{i\mathbf{k}_{31} \cdot \mathbf{r}_{12}} = e^{(3/2)i(\mathbf{k}_1 \cdot \mathbf{x}_2 + \mathbf{k}_2 \cdot \mathbf{x}_3 + \mathbf{k}_3 \cdot \mathbf{x}_1)}, \quad (7d)$$

$$A(5) = \psi_{\mathbf{k}_1\mathbf{k}_2}^C(\mathbf{x}_3, \mathbf{x}_1) \psi_{\mathbf{k}_2\mathbf{k}_3}^C(\mathbf{x}_1, \mathbf{x}_2) \psi_{\mathbf{k}_3\mathbf{k}_1}^C(\mathbf{x}_2, \mathbf{x}_3) \xrightarrow{\text{PW}} e^{i\mathbf{k}_{12} \cdot \mathbf{r}_{31}} e^{i\mathbf{k}_{23} \cdot \mathbf{r}_{12}} e^{i\mathbf{k}_{31} \cdot \mathbf{r}_{23}} = e^{(3/2)i(\mathbf{k}_1 \cdot \mathbf{x}_3 + \mathbf{k}_2 \cdot \mathbf{x}_1 + \mathbf{k}_3 \cdot \mathbf{x}_2)}, \quad (7e)$$

$$A(6) = \psi_{\mathbf{k}_1\mathbf{k}_2}^C(\mathbf{x}_3, \mathbf{x}_2) \psi_{\mathbf{k}_2\mathbf{k}_3}^C(\mathbf{x}_2, \mathbf{x}_1) \psi_{\mathbf{k}_3\mathbf{k}_1}^C(\mathbf{x}_1, \mathbf{x}_3)$$

$$\xrightarrow{\text{PW}} e^{i\mathbf{k}_{12}\cdot\mathbf{r}_{32}} e^{i\mathbf{k}_{23}\cdot\mathbf{r}_{21}} e^{i\mathbf{k}_{31}\cdot\mathbf{r}_{13}} = e^{(3/2)i(\mathbf{k}_1\cdot\mathbf{x}_3+\mathbf{k}_2\cdot\mathbf{x}_2+\mathbf{k}_3\cdot\mathbf{x}_1)} . \quad (7f)$$

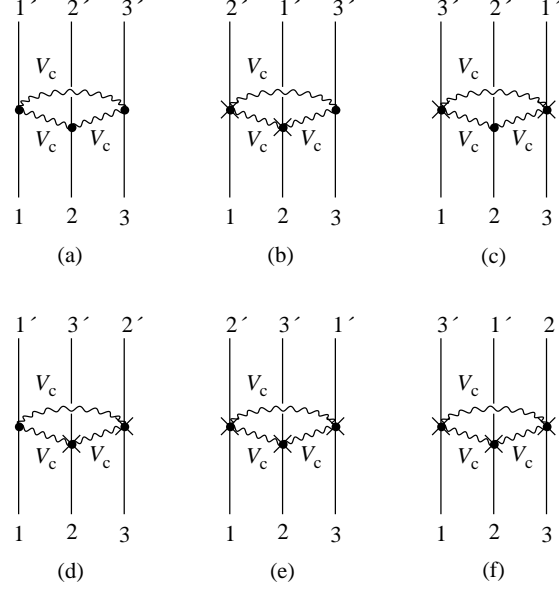


Fig. 2. Diagram reflecting three-charged particles Bose-Einstein Correlation (BEC) and Coulombic potential ( $V_c$ ).  $\times$  means the exchange effect of BEC.

Notice that, except for the factor  $3/2$ , exponential functions are the same expressions as those present in the integrand of Eq. (2). This difference is attributed to the fact that Coulomb wave function used here describes two-charged particles collisions, therefore factor  $3/2$  appears because there are relevant two-particle three combinations among three-charged particles.

Combining Eqs. (7) and Figs. 2, we obtain the following three sets of equations:

$$F_1 = \frac{1}{6} \sum_{i=1}^6 A(i) A^*(i) \xrightarrow{\text{PW}} 1 , \quad (8a)$$

$$F_2 = \frac{1}{6} [A(1)A^*(2) + A(1)A^*(3) + A(1)A^*(6) + A(2)A^*(4) + A(2)A^*(5) + A(3)A^*(4) + A(3)A^*(5) + A(4)A^*(6) + A(5)A^*(6) + \text{c. c.}] \xrightarrow{\text{PW}} \text{BEC between two-charged particles} \quad (8b)$$

(See Figs. 2 (b)~(d)) ,

$$F_3 = \frac{1}{6} [A(1)A^*(4) + A(1)A^*(5) + A(2)A^*(3) + A(2)A^*(6) + A(3)A^*(6) + A(4)A^*(5) + \text{c. c.}] \xrightarrow{\text{PW}} \text{BEC among three-charged particles} \quad (8c)$$

(See Figs. 2 (e) and (f)) .

Combining now Eqs. (8) and the concept of partial coherence for the BEC [17,20], we can introduce a coherence parameter  $\sqrt{\lambda}$  for the single mark ( $\times$ ) in Fig. 2. The  $\lambda = 1$  corresponds to the totally chaotic source, which is the assumption behind Eq. (5). Taking into account the strength of the degree of coherence  $\lambda$  between two-identical bosons and  $\lambda^{3/2}$  among three-identical bosons in Figs. 2, we can finally express the BEC for three identical charged particles as:

$$\frac{N^{(3+ \text{ or } 3-)}}{N^{BG}} \cong C \int d^3\mathbf{x}_1 \rho(\mathbf{x}_1) d^3\mathbf{x}_2 \rho(\mathbf{x}_2) d^3\mathbf{x}_3 \rho(\mathbf{x}_3) [F_1 + \lambda F_2 + \lambda^{3/2} F_3] . \quad (9)$$

Equation (9) is the modified theoretical formula we were looking for. It differs from Eq. (5) originally proposed by Alt et. al. in [5] by the presence of the degree of coherence  $\lambda$  and in the limit of  $\eta_{ij} \rightarrow 0$  it becomes

$$\text{Eq. (9)} \xrightarrow{\eta_{ij} \rightarrow 0} C \left( 1 + 3\lambda e^{-\frac{3}{4}R^2 Q_3^2} + 2\lambda^{3/2} e^{-\frac{9}{8}R^2 Q_3^2} \right) , \quad (10)$$

which is the extended formula proposed some time ago by Deutschmann et al. [20].

It should be noticed that Eq. (9) can be applied to data corrected only by the Gamow factor  $G(\eta_{12})G(\eta_{23})G(\eta_{31})$  in an ideal case [21], because Eq. (9) is described by the Coulomb wave functions including the Gamow factors (see Ref. [22])<sup>5</sup>.

#### 4 Reanalyses of NA44 Collaboration data by means of Eq. (9)

At present we have no data corrected only by the Gamow factors, therefore we apply Eq. (9) to the analysis of NA44 Collaboration data [17] using the CERN-MINUIT program. Our results are shown in Fig. 3 and Table 2. Comparing them with those of Table 1, it can be said that the  $\chi^2$ -value becomes smaller, i.e., the agreement is now better. The range of interaction becomes also smaller. For the sake of reference we present in Table 2 also results obtained by using Eq. (10).

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<sup>5</sup> In other words, ideal data sets for Eq. (9) are of the form of

$$\text{Corrected data} = (\text{raw data}) \times K_{\text{spc}} \times K_{\text{acceptance}} \times K_{\text{Gamow}} ,$$

where  $K_{\text{Gamow}} = 1/(G(\eta_{12})G(\eta_{23})G(\eta_{31}))$ .

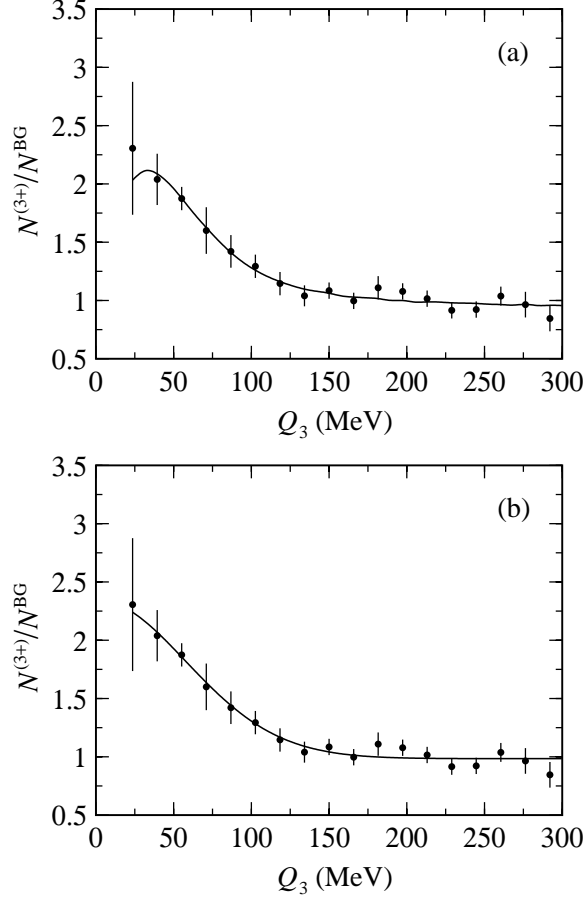


Fig. 3. Reanalyses of  $3\pi^+$  BEC in S + Pb collision [17]. (a) is result of Eq. (9). (b) is that of Eq. (10).

Table 2

Reanalyses of  $3\pi^+$  BEC in S + Pb collision [17] by Eqs. (9) and (10).

Formula	$C$	$R$ [fm]	$\lambda$	$\chi^2/N_{dof}$
Eq. (9)	$0.917 \pm 0.032$	$1.53 \pm 0.20$	$0.55 \pm 0.07$	6.7/15
Eq. (10)	$0.984 \pm 0.029$	$2.60 \pm 0.28$	$0.33 \pm 0.04$	7.7/15

From results of Table 2, we see that  $R = 1.53$  fm by Eq. (9) is small. The reason is attributed to the fraction of partially coherent effect ( $\lambda$ ). From comparisons between results by Eqs. (9) and (10), it can be seen the interaction region becomes smaller, and the degree of coherence does conversely bigger due to the Coulomb interaction.

## 5 Possible interpretation of $\lambda$ by core-halo model

The core-halo model has been proposed by Csörgő et al. [18]. We study whether or not there is physical connection between our previous formulation and theirs [18]. To apply their model to Eq. (8), first of all we use the following source functions

$$\rho(x_i) = \rho_c(x_i) + \rho_{halo}(x_i) , \quad (11)$$

where  $\rho_c(x_i) = \frac{1}{(2\pi R^2)^{3/2}} \exp\left[-\frac{\mathbf{x}_i^2}{2R^2}\right]$  and  $\rho_{halo}(x_i) = \frac{1}{(2\pi R_h^2)^{3/2}} \exp\left[-\frac{\mathbf{x}_i^2}{2R_h^2}\right]$ .

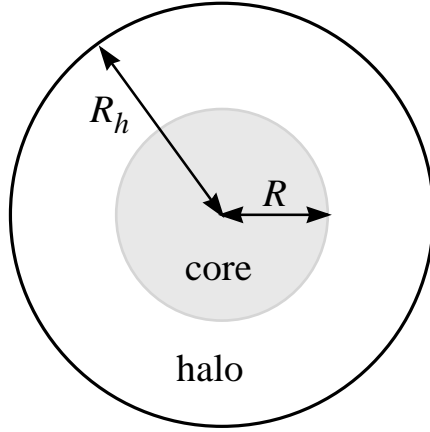


Fig. 4. Physical picture of core-halo model.

In concrete analyses, the radius of the halo becomes to be infinite, due to the effect of long-lived resonances ( $R_h \rightarrow \infty$ ). The exchange function of the halo part due to the BEC,  $E_{halo} = \exp(-Q^2 R_h^2) \rightarrow 0$ . Then we can interpret  $\lambda$  in Eq. (9) by their parameter, i. e., fraction of multiplicity from the core part,  $f_c = \langle n_{core} \rangle / \langle n_{tot} \rangle$ , as  $\lambda = f_c^2$ . In this case, it should be stressed that their parameter  $p_c = \langle n_{co} \rangle / \langle n_{core} \rangle$ , the fraction of the coherently produced multiplicity from the core part is zero.

Moreover, we can assume the laser optical approach for the core part [18], introducing a parameter  $p = \langle n_{chao} \rangle / \langle n_{core} \rangle$  [1]. Notice that  $p = 1 - p_c$  [18]. For the cross mark ( $\times$ ) in Fig. 2, we assume two components of coherently ( $1 - p$ ) and chaotically produced particles ( $p$ ). Then we obtain the following expression

$$\begin{aligned} \frac{N^{(3+ \text{ or } 3-)}}{N^{BG}} \cong C \bigg[ & \int d^3 \mathbf{x}_1 \rho_c(\mathbf{x}_1) d^3 \mathbf{x}_2 \rho_c(\mathbf{x}_2) d^3 \mathbf{x}_3 \rho_c(\mathbf{x}_3) (F_1 + f_c^2 p^2 F_2 + f_c^3 p^3 F_3) \\ & + \int d^3 \mathbf{x}_1 \rho_c(\mathbf{x}_1) \cdot \delta^3(\mathbf{x}_1) d^3 \mathbf{x}_2 \rho_c(\mathbf{x}_2) d^3 \mathbf{x}_3 \rho_c(\mathbf{x}_3) \end{aligned}$$

$$\cdot \left( 2f_c^2 p(1-p)F_2 + 3f_c^3 p^2(1-p)F_3 \right) \Big] . \quad (12)$$

The effective degree of coherence,  $\lambda_3^*$ , the intercept at smallest  $Q_3$  is given as

$$\lambda_3^* = f_c^2(p^2 + 2p(1-p)) + f_c^3(p^3 + 3p^2(1-p)) . \quad (13)$$

In Eq. (12), as  $p = 1$ , we obtain Eq. (9) with  $\lambda = f_c^2$ . On the contrary, as  $f_c = 1$ , we obtain an expression of laser optical approach [1]. By making use of Eq. (12) and an expression for the two-charged particles of the BEC, we can analyse the data by NA44 Collaboration.

For the two-charged particles of the BEC by means of the core-halo model with laser optical approach, we have the following equation,

$$\frac{N^{(2+ \text{ or } 2-)}}{N^{BG}} \cong C \left[ \int d^3\mathbf{x}_1 \rho(\mathbf{x}_1) d^3\mathbf{x}_2 \rho(\mathbf{x}_2) (G_1 + f_c^2 p^2 G_2) \right. \\ \left. + \int d^3\mathbf{x}_1 \rho(\mathbf{x}_1) d^3\mathbf{x}_2 \rho(\mathbf{x}_2) \cdot \delta^3(\mathbf{x}_2) 2f_c^2 p(1-p) G_2 \right] , \quad (14)$$

where  $G_1 = \frac{1}{2} \left( \left| \psi_{\mathbf{k}_1 \mathbf{k}_2}^C(\mathbf{x}_1, \mathbf{x}_2) \right|^2 + \left| \psi_{\mathbf{k}_1 \mathbf{k}_2}^C(\mathbf{x}_2, \mathbf{x}_1) \right|^2 \right)$  and  $G_2 = \text{Re} \left( \psi_{\mathbf{k}_1 \mathbf{k}_2}^{C*}(\mathbf{x}_1, \mathbf{x}_2) \cdot \psi_{\mathbf{k}_1 \mathbf{k}_2}^C(\mathbf{x}_2, \mathbf{x}_1) \right)$ . The effective degree of coherence  $\lambda_2^*$  is given as

$$\lambda_2^* = f_c^2(p^2 + 2p(1-p)) . \quad (15)$$

Results of our analyses are shown in Table 3 and Figure 5. As is seen in them, the interaction ranges of  $R(\text{core part})$  are estimated in the ranges of  $1.5 \text{ fm} < R(\text{core}) < 1.8 \text{ fm}$ , and  $4.7 \text{ fm} < R(\text{core}) < 5.4 \text{ fm}$ , for the BEC of  $3\pi$  and  $3\pi \rightarrow 2\pi$ , respectively.

The common region between results of the BEC of three-pion and two-pion is roughly described by  $0.6 \lesssim p \lesssim 1.0$  and  $f_c \sim 0.7$ .

## 6 Concluding remarks

We have derived the theoretical formula for the BEC of three-charged identical particle using both the Coulomb wave functions and the notion of the degree of coherence and compared it with the experimental data <sup>6</sup>. Historically the

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<sup>6</sup> For the numerical calculations of Eqs. (5) and (9) (in order to save the CPU-time), we have first calculated  $10^3$  k values of the Coulomb wave functions, which were then

Table 3

Typical results from analyses of data of NA44 Collaboration by Eqs. (12)~(15).

$3\pi$ BEC			
$p$	1.0	0.8	0.6
$f_c$	$0.743 \pm 0.050$	$0.779 \pm 0.055$	$0.856 \pm 0.061$
$\lambda_3^*$	0.964	1.007	1.022
$R$ (fm)	$1.53 \pm 0.21$	$1.72 \pm 0.25$	$1.87 \pm 0.27$
$\chi^2/N_{dof}$	6.7/15	6.6/15	6.6/15
$3\pi \rightarrow 2\pi$ BEC			
$p$	1.0	0.8	0.6
$f_c$	$0.633 \pm 0.028$	$0.647 \pm 0.029$	$0.688 \pm 0.032$
$\lambda_2^*$	0.400	0.402	0.398
$R$ (fm)	$4.69 \pm 0.45$	$5.34 \pm 0.54$	$5.85 \pm 0.59$
$\chi^2/N_{dof}$	14.6/17	14.9/17	14.9/17

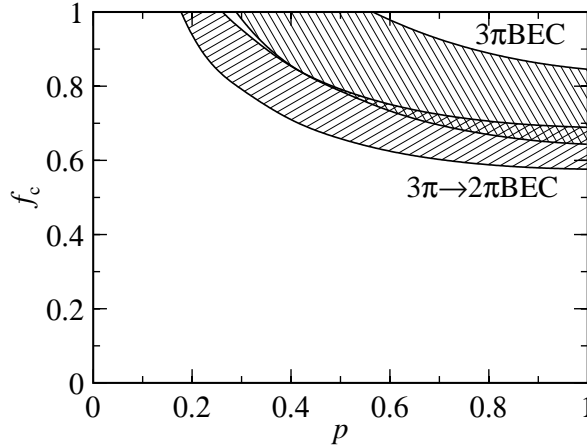


Fig. 5. Sets of  $f_c$  and  $p$  estimated in analyses of BEC of two-charged and three-charged pions using the CERN-MINUIT program. Widths of  $f_c$ 's stand for error bar of  $\pm 2\sigma$ .

degree of coherence in the BEC of the two-identical bosons has been introduced by experimentalist [20], and theoretical works in this direction have been performed in Ref. [9].

Our first analyses suggest that the degree of coherence  $\lambda$  is a necessary ingredient also for the BEC of three-charged particles, in the same way as it was for the BEC for two-charged identical particles. This fact means that the

used together with some interpolation procedure during the concrete calculations. In this way we could make use of the CERN-MINUIT program in our analyses.

source producing finally observed particles is not purely chaotic. However the investigation in 5th paragraph elucidates that our modified formulation can be interpreted by the core-halo model. Our degree of coherence  $\lambda$  is equal to  $f_c^2$  [introduced in Ref. [18]], provided that particles are chaotically produced;  $\lambda = f_c^2 (= (\langle n_{core} \rangle / \langle n_{tot} \rangle)^2)$ . Moreover, if we can assume the laser-optical approach for the cross mark ( $\times$ ) in Fig. 2, we obtain Eqs. (12)  $\sim$  (15). By making use of them, we obtain Table 3 and Fig. 5. There is common region among results from analyses of the BEC of  $3\pi$  and  $3\pi \rightarrow 2\pi$  processes,  $0.6 \lesssim p \lesssim 1.0$  and  $f_c \sim 0.7$ . However, the magnitude of the interaction regions estimated by the BEC of  $3\pi$  and  $3\pi \rightarrow 2\pi$  process are different. This problem should be considered in the future.

It should be noticed that also recent data on the BEC of  $3\pi^-$  reported by OPAL Collaboration [23] suggest the necessity of introduction of some degree of coherence or  $f_c$  in Ref. [18] <sup>7</sup>.

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<sup>7</sup> Analysis of these data will be reported elsewhere [24].

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## Addendum

Because we had no sufficient information on the raw data and corrected data in 2001, we utilized the corrected data with  $K_{\text{Coulomb}}$  in our analyses. Of course we mentioned that our Eqs. (9), (12), and (14) are available for the corrected data including  $K_{\text{SPC}}$  (single particle correction (SPC)) and  $K_{\text{acceptance}}$ , explicitly. Referring to Refs. [25,26], we have examined the methods of correction used by NA44 Collaboration [17]. In this addendum, thus we can present the data including  $K_{\text{SPS}}$  and  $K_{\text{acceptance}}$  and analyze them by means of Eqs.(9), (12) and (14). Before concrete analyses, we categorize raw and corrected data in Table 4.

Table 4

Category of data. Notice that  $R = \text{input}$  is necessary for  $K_{\text{Coulomb}}$ , where  $R = 5 \text{ fm}$  is used in Refs. [17,25,26].

1) Raw data	raw data
2) Quasi-corrected data (Q-CD)	$(\text{raw data}) \times K_{\text{SPC}} \times K_{\text{acceptance}}$
3) Corrected data with Gamow	$(\text{raw data}) \times K_{\text{SPC}} \times K_{\text{acceptance}} \times K_{\text{Gamow}}$
4) Corrected data with Coulomb	$(\text{raw data}) \times K_{\text{SPC}} \times K_{\text{acceptance}} \times K_{\text{Coulomb}}$

Using the data in Ref. [17] with  $K_{\text{Coulomb}}$  or  $K_{\text{Gamow}}$ , we can obtain the quasi-corrected data (Q-CD) shown in Fig. 6, which are available for our purpose. Using Eqs. (5) and (9), we obtain the results shown in Table 5. As seen in Fig. 6, the our formulation including the degree of coherence  $\lambda$ , i.e., Eq. (9), seems to be available for analyses of the charged  $\pi$  BEC.

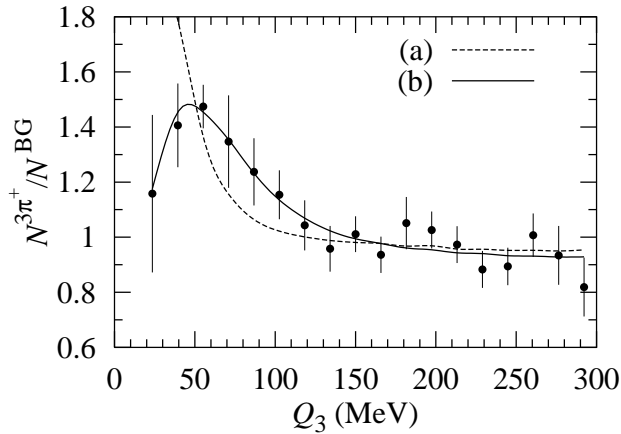


Fig. 6. Analyses of quasi-corrected data (Q-CD). (a) and (b) are results of Eqs. (5) and (9), respectively. .

To examine availability of the core-halo model described by the Coulomb wave functions, i.e., Eqs. (12) and (14), we analyze the quasi-corrected data (Q-CD).

Table 5

Estimated values for the data [17] by Eqs. (5) and (9) using CERN-MINUIT program.

Formulas	$C$	$R$ [fm]	$\lambda$	$\chi^2/N_{dof}$
Eq. (5)	$0.94 \pm 0.03$	$5.83 \pm 0.40$	—	30/16
Eq. (9)	$0.91 \pm 0.03$	$2.89 \pm 0.39$	$0.45 \pm 0.05$	6.7/15

Our results are shown in Fig. 7 and Table 6. Coincidences between  $3\pi$  BEC and  $3\pi \rightarrow 2\pi$  BEC in Fig. 7 ( $f_c$  vs.  $p$ ) seem to be more enlarged than those of Fig. 5.

In this addendum, we have analyzed the quasi-corrected data (Q-CD) named in Table 4, utilizing Eqs. (5), (6), (12) and (14). It can be said that our formulation works well for the description of the charged  $3\pi$  BEC.

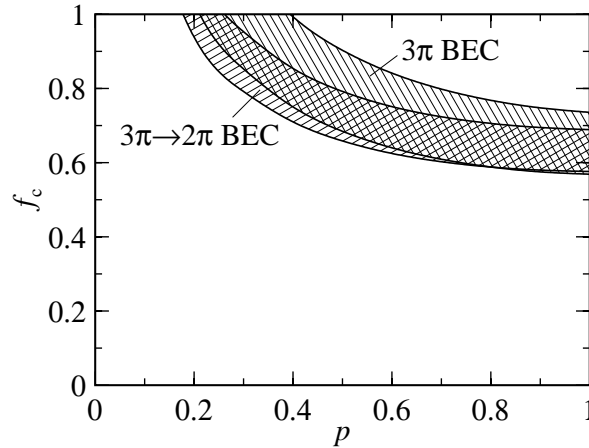


Fig. 7. Sets of  $f_c$  and  $p$  estimated in analyses of BEC of two-charged and three-charged pions using the CERN-MINUIT program. Widths of  $f_c$ 's stand for error bar of  $\pm 2\sigma$ .

As seen in the explanation above, Fig. 1(a) and the upper line (Eq. (5)) of Table 1, and Fig. 3(a) and the upper line (Eq. (9)) of Table 2 should be replaced by Fig. 6 and Table 5. Moreover, Fig. 5 and Table 4 should be replaced by Fig. 7 and Table 6.

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Table 6

Typical results from analyses of data of NA44 Collaboration by Eqs. (12)~(15).  $R_{\text{plane}} = (3/2)R$ , where  $R_{\text{plane}}$  is corresponding to a parameter in the plane wave formulation.

$p$	1.0	0.8	0.6
$3\pi$ BEC			
$f_c$	$0.67 \pm 0.04$	$0.70 \pm 0.04$	$0.76 \pm 0.05$
$\lambda_3^*$	0.75	0.77	0.78
$R$ (fm)	$2.89 \pm 0.39$	$3.22 \pm 0.46$	$3.47 \pm 0.51$
$R_{\text{plane}}$ (fm)	$4.33 \pm 0.58$	$4.83 \pm 0.69$	$5.21 \pm 0.76$
$\chi^2/N_{\text{dof}}$	6.7/15	6.8/15	6.8/15
$3\pi \rightarrow 2\pi$ BEC			
$f_c$	$0.67 \pm 0.03$	$0.69 \pm 0.03$	$0.73 \pm 0.03$
$\lambda_2^*$	0.45	0.46	0.45
$R$ (fm)	$4.42 \pm 0.38$	$5.01 \pm 0.45$	$5.49 \pm 0.50$
$\chi^2/N_{\text{dof}}$	14.5/17	14.7/17	14.7/17

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